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On the boundary structure of the convex hull of random points

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Dedicated to Peter M. Gruber on the occasion of his 70th birthday

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Abstract. It is well known that the vertices of the convex hull of n random points, which are chosen independently and uniformly from the interior of a convex polygon, are concentrated in the neighbourhoods of the vertices of the polygon as n tends to infinity. Here concentration means that the number of vertices of the convex hull outside of the neighbourhoods of the vertices of the polygon is negligible asymptotically. The first moment of the number of vertices of the convex hull in a neighbourhood of a vertex of the polygon was obtained in 1963 by Rényi and Sulanke in a classical paper. The second moment was achieved in 1988 by Groeneboom using a Poisson point process approximation technique. Due to the complexity of the occurring calculations the extension of this technique to higher moments appears to be out of reach. Based on a purely geometric approach, which avoids stochastic processes, we derive the moments of all orders.

Key words. Random points, convex hull, higher moments.

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1 Introduction

When investigating the number of vertices of the convex hull of n points chosen independently and uniformly from the interior of a convex polygon, it is essential to observe that the vertices are concentrated close to the vertices of the polygon as n tends to infinity. In order to see how the classical paper from 1963 by Rényi and Sulanke [19] reflects this fact, it is illuminating to recall their approach.

Obviously, the number of vertices of the convex hull is identical to the number of edges of the convex hull. Any two random points are the endpoints of an edge of the convex hull if all remaining $n-2$ random points lie on one and the same side of the line containing the edge. As the points are identically distributed, the probability of this event is the same for any selection of two random points. Hence we are led to the question of determining the probability that a line through two points is such that $n-2$ random points are on the same side. This probability is calculated for all lines, distinguishing which edges of the given polygon a line intersects. If the occurring values are multiplied by the number of selections of 2 points out of n, the result for two neighbouring edges, independent of the angle between the edges and the lengths of the edges, is $2/3 \log n + O(1)$ as n tends to infinity. Notice the invariance under non-singular affine transformations. The result for non-neighbouring edges is $O(1)$ if there is one further edge between the intersected edges and $o(1)$ if there are two or more edges between. A second look at the calculations shows that the value $2/3 \log n + O(1)$ remains unchanged if instead of all lines intersecting two neighbouring edges only those lines are considered, which intersect the edges in an arbitrary small neighbourhood of the vertex, where the two neighbouring edges meet.

Clearly, the probability that two random points contribute an edge to the convex hull, multiplied by the number of selections of 2 points out of n, gives just the *expected value* of the number of edges of the convex hull and, equivalently, the *expected* number of vertices of the convex hull.

It does not appear promising to extend the sketched classical approach by Rényi and Sulanke to the second moment. In spite of many efforts, cf., e.g., the work of Jewell and Romano [15, p. 547], [16, p. 424], essentially no progress was achieved for a quarter of a century. Then, in 1988, Groeneboom [13] came up with the idea of approximating the process of those vertices of the convex hull of a uniform sample of random points in the unit square with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, which are close to the origin $(0,0)$, by the process of vertices of the "left-lower" boundary of the convex hull of a realization of a Poisson point process on $\mathbb{R}_+^2$ with intensity n times Lebesgue measure. It turns out that the variance of the number of vertices of the latter process can indeed be calculated, even though the necessary effort is tremendous. It can then be deduced ([13, p. 328]; also cf. [7, Section 5.4]) that the variance of the number of vertices of the convex hull of the uniform sample, which are close to the origin, is asymptotically equal to $10/27 \log n$.

Considering the calculations required to derive the variance by Groeneboom's approach, it appears to be hopeless to extend the method in order to obtain third or higher order information.

Recalling the invariance under non-singular affine transformations, we investigate in the present paper the number of those vertices of the convex hull of a uniform sample of random points in the triangle given by $(0,1)$, $(0,0)$, and $(1,0)$, which are situated in a neighbourhood of the origin $(0,0)$. These vertices can be identified in the following way: Take a line intersecting both the positive x- and the positive y-axis, and consider the convex hull of the two intersection points and those of the points of the sample, which lie on the same side of the line as the origin. The influence of the position of the line on the number of identified vertices is negligible asymptotically. The line may even be taken through the points $(0,1)$ and $(1,0)$. (See [3]; also cf. [10].)

Therefore our task is the investigation of the random variable N_n defined as follows: Assume that n points $P_1, \dots, P_n$ are distributed independently and uniformly in the interior of the triangle with vertices $(0,1)$, $(0,0)$, and $(1,0)$. Consider the convex hull of $(0,1), P_1, \dots, P_n$, and $(1,0)$. Denote by N_n the number of those of the points $P_1, \dots, P_n$, which are vertices of the convex hull.

Based on a simple geometric idea, for the probabilities $p_k^{(n)}$ $(k = 1, \dots, n)$ that $N_n = k$ an explicit formula was obtained in [8]:

$$p_k^{(n)} = 2^k \sum \frac{i_1 \dots i_k}{i_1(i_1+1)(i_1+i_2)(i_1+i_2+1)\dots(i_1+\dots+i_k)(i_1+\dots+i_k+1)},$$

where the sum is taken over all $i_1, \dots, i_k \in \mathbb{N}$ such that $i_1 + \dots + i_k = n$. (The result was announced in [6].) The crucial point now consists in deriving formulae for the moments of N_n, from which their asymptotic behaviour as n tends to infinity becomes apparent.

As a first step a linear second order difference equation for the m-th moment of N_n is established (Theorem 1). Its right hand side

$$E(N_{n-1}+1)^m - EN_{n-1}^m = \sum_{j=0}^{m-1} \binom{m}{j} EN_{n-1}^j$$

involves the moments EN_{n-1}^j of order $j = 1, \dots, m-1$. Once the moments up to order $m-1$ are obtained, the m-th moment follows as the solution of the difference equation.

As a second step the difference equation is dealt with. Remarkably, the coefficients in the equation for the m-th moment do not depend on m, i.e., the homogenous equation is the same for all moments. We solve the difference equation for any right hand side, say $g(n)$, thus obtaining the solution of the equation in terms of $g(n)$. Choosing $g(n)$ to be $E(N_{n-1}+1)^m - EN_{n-1}^m$, we obtain the m-th moment of N_n in terms of the moments of smaller order (Theorem 2).

The classical result by Rényi and Sulanke for EN_n now follows just by putting $g(n) = 1$ (Corollary 1). Then putting $g(n) = 2EN_{n-1} + 1$ yields Groeneboom's result for the variance (Corollary 2; also cf. [10]). The results for EN_n and EN_n^2 imply EN_n^3 (Corollary 3). We also state the formula for EN_n^4 following from the first three moments (Corollary 4). Continuing in this way, any further moment can be derived.

Corollaries 1 to 4 show for $m = 1, 2, 3$, and 4 that

$$EN_n^m = \left(\frac{2}{3}\log n\right)^m + O\big(\log^{m-1} n\big)$$

as n tends to infinity. The precise asymptotic behaviour of the weight function $w_{k,n}$ occurring in Theorem 2 is finally used to prove the formula for moments of any order m (Theorem 3).

The present definition of N_n turns out not only to be suitable to obtain the required asymptotic formulae, but also to be exactly the right one in order to obtain explicit formulae for the number of vertices of the convex hull of a fixed number of random points; see [8, p. 248] and the announcement [9].

For more information about the convex hull of random points see in particular the books by Mathai [17] and Schneider and Weil [23], as well as the surveys by Affentranger [1], Bárány [2], Buchta [5], Gruber [14], Reitzner [18], Schneider [20], [21], [22], and Weil and Wieacker [24]. Many references are also contained in [3], [7], [11], and [12]. Important recent results concerning the variance in the higher dimensional case are due to Bárány and Reitzner [4].

2 Formulae for the m-th moment

Theorem 1. *For any $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ the m-th moment of N_n is the (unique) solution of the second order difference equation*

$$\frac{n(n+1)}{2} EN_n^m - (n^2 - n + 1) EN_{n-1}^m + \frac{(n-2)(n-1)}{2} EN_{n-2}^m$$
$$= E(N_{n-1}+1)^m - EN_{n-1}^m$$

with initial values $EN_1^m = 1$ and $EN_2^m = (2^m + 2)/3$.

Proof. We make use of the formula for the probability $p_k^{(n)}$ $(k = 1, \ldots, n)$ that $N_n = k$, which is stated in the introduction. In this formula the index i_k is at least 1 and at most $n - k + 1$. Correspondingly, the sum of the remaining indices $i_1, \ldots, i_{k-1}$ is at least $k - 1$ and at most $n - 1$. Denoting this sum by j and arranging the formula according to the values of j, we see that $p_k^{(n)}$ can be broken down to $p_{k-1}^{(j)}$ $(j = k-1, \ldots, n-1)$:

$$p_k^{(n)} = \frac{2}{n(n+1)} \sum_{j=k-1}^{n-1} (n-j) p_{k-1}^{(j)},$$

where $p_0^{(0)} = 1$ and $p_0^{(j)} = 0$ for $j \in \mathbb{N}$. From

$$\frac{n(n+1)}{2} p_k^{(n)} = \sum_{j=k-1}^{n-1} (n-j) p_{k-1}^{(j)}$$

it follows that

$$\frac{n(n+1)}{2} p_k^{(n)} - \frac{(n-1)n}{2} p_k^{(n-1)} = \sum_{j=k-1}^{n-1} p_{k-1}^{(j)}$$

and that

$$\left(\frac{n(n+1)}{2} p_k^{(n)} - \frac{(n-1)n}{2} p_k^{(n-1)} \right)$$
$$- \left(\frac{(n-1)n}{2} p_k^{(n-1)} - \frac{(n-2)(n-1)}{2} p_k^{(n-2)} \right) = p_{k-1}^{(n-1)}.$$

The resulting identity

$$\frac{n(n+1)}{2} p_k^{(n)} - (n-1)n p_k^{(n-1)} + \frac{(n-2)(n-1)}{2} p_k^{(n-2)} = p_{k-1}^{(n-1)},$$

first multiplied by k^m on the left hand side and by $((k-1)+1)^m$ on the right hand side $(k=1,\dots,n)$ and then summed up from $k=1$ to $k=n$, yields

$$\frac{n(n+1)}{2} EN_n^m - (n-1)nEN_{n-1}^m + \frac{(n-2)(n-1)}{2} EN_{n-2}^m = E(N_{n-1}+1)^m.$$

Hence the claimed difference equation follows immediately. Furthermore, $p_1^{(1)} = 1$ such that $EN_1^m = 1$ and $p_1^{(2)} = 2/3$, $p_2^{(2)} = 1/3$ such that $EN_2^m = (2^m+2)/3$. □

Theorem 2. *For any $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ the m-th moment of N_n is determined by the j-th moments $(j = 1,\dots,m-1)$ of N_k $(k=1,\dots,n-1)$ according to the identity*

$$EN_n^m = 1 + \sum_{k=1}^{n-1} w_{k,n}\big(E(N_k+1)^m - EN_k^m\big)$$

with weights

$$w_{k,n} = 2k(k+1)\sum_{i=k+1}^{n} \frac{1}{(i^2-1)i^2}.$$

Proof. Noticing that in the difference equation in Theorem 1 the coefficients on the left hand side do not depend on m, we consider the difference equation

$$\frac{n(n+1)}{2} f(n) - (n^2-n+1)f(n-1) + \frac{(n-2)(n-1)}{2} f(n-2) = g(n),$$

where $f(n)$ is the unknown function and $g(n)$ a given right hand side. This equation can also be written in the form

$$\frac{n(n+1)}{2}(f(n)-f(n-1)) - \frac{(n-2)(n-1)}{2}(f(n-1)-f(n-2)) = g(n)$$

and hence in the form

$$f(n)-f(n-1) = \frac{(n-2)(n-1)}{n(n+1)}(f(n-1)-f(n-2)) + \frac{2}{n(n+1)} g(n).$$

Iteration yields

$$f(n)-f(n-1) = \frac{12}{(n^2-1)n^2}(f(2)-f(1)) + \frac{2}{(n^2-1)n^2}\sum_{k=3}^{n}(k-1)k\, g(k).$$

Therefore the solution of the difference equation is given by

$$f(n) = f(1) + \sum_{i=2}^{n} \frac{2}{(i^2-1)i^2}\left(6(f(2)-f(1)) + \sum_{k=3}^{i}(k-1)k\, g(k)\right).$$

Actually we have $f(1) = EN_1^m = 1$ and $f(2) - f(1) = EN_2^m - EN_1^m = (2^m - 1)/3$ according to Theorem 1. Furthermore, $g(2) = E(N_1 + 1)^m - EN_1^m = 2^m - 1$ since $N_1 \equiv 1$. Thus

$$\begin{aligned} EN_n^m &= 1 + \sum_{i=2}^{n} \frac{2}{(i^2-1)i^2} \sum_{k=2}^{i} (k-1)k(E(N_{k-1}+1)^m - EN_{k-1}^m) \\ &= 1 + \sum_{i=2}^{n} \frac{2}{(i^2-1)i^2} \sum_{k=1}^{i-1} k(k+1)(E(N_k+1)^m - EN_k^m) \\ &= 1 + \sum_{k=1}^{n} 2k(k+1)(E(N_k+1)^m - EN_k^m) \sum_{i=k+1}^{n} \frac{1}{(i^2-1)i^2}, \end{aligned}$$

as claimed. □

Corollary 1. *The first moment of N_n is given by*

$$EN_n = \frac{2}{3} H_n + \frac{1}{3},$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Proof. According to Theorem 2

$$EN_n = 1 + \sum_{k=1}^{n-1} w_{k,n} = 1 + \sum_{k=1}^{n-1} 2k(k+1) \sum_{i=k+1}^{n} \frac{1}{(i^2-1)i^2}.$$

Interchanging the order of summation yields

$$EN_n = 1 + 2 \sum_{i=2}^{n} \frac{1}{(i^2-1)i^2} \sum_{k=1}^{i-1} k(k+1) = \frac{2}{3} H_n + \frac{1}{3},$$

since $\sum_{k=1}^{i-1} k(k+1) = \frac{1}{3}(i-1)i(i+1)$. □

Starting with the first moment $EN_n = \frac{2}{3} H_n + \frac{1}{3}$, a repeated application of Theorem 2 yields a representation of the m-th moment in terms of generalized harmonic sums

$$\sum_{k_1=1}^{n} \frac{1}{k_1^{s_1}} \sum_{k_2=1}^{k_1} \frac{1}{k_2^{s_2}} \cdots \sum_{k_t=1}^{k_{t-1}} \frac{1}{k_t^{s_t}},$$

with $s_1, \ldots, s_t \in \mathbb{N}$, for which we introduce the notation $H_n^{(s_1,\ldots,s_t)}$. Instead of $H_n^{(1)}$ we simply write H_n. In general, the representation is not unambiguous, e.g. $H_n^{(1,1)} = \frac{1}{2} H_n^2 + \frac{1}{2} H_n^{(2)}$. Moreover, any generalized harmonic sum can be expressed as a polynomial of generalized harmonic sums, which are either just H_n or such that $s_1 \geq 2$ and $s_1 \geq s_2 \geq \cdots \geq s_t$. This is easy to see by interchanging the order of summation, e.g.

$$\begin{aligned} H_n^{(1,2)} &= \sum_{k_1=1}^{n} \frac{1}{k_1} \sum_{k_2=1}^{k_1} \frac{1}{k_2^2} = \sum_{k_2=1}^{n} \frac{1}{k_2^2} \sum_{k_1=k_2}^{n} \frac{1}{k_1} \\ &= \sum_{k_2=1}^{n} \frac{1}{k_2^2} \left(\sum_{k_1=1}^{n} \frac{1}{k_1} - \sum_{k_1=1}^{k_2} \frac{1}{k_1} + \frac{1}{k_2} \right) \\ &= H_n H_n^{(2)} - H_n^{(2,1)} + H_n^{(3)}. \end{aligned}$$

Clearly, the asymptotic behaviour of H_n is given by $H_n = \log n + O(1)$, whereas any $H_n^{(s_1,\ldots,s_t)}$ with $s_1 \geq 2$ tends to a constant as n tends to infinity. Therefore, the asymptotic behaviour of EN_n^m, once expressed by generalized harmonic sums of the mentioned type, is obvious at first sight. In the subsequent corollaries we state the arising formulae for $m = 2, 3$, and 4.

Corollary 2. *The second moment of N_n is given by*

$$EN_n^2 = \frac{4}{9} H_n^2 + \frac{22}{27} H_n + \frac{4}{9} H_n^{(2)} - \frac{25}{27} + \frac{4}{9} \frac{1}{n+1}$$

and the variance of N_n by

$$\operatorname{var} N_n = \frac{10}{27} H_n + \frac{4}{9} H_n^{(2)} - \frac{28}{27} + \frac{4}{9} \frac{1}{n+1}.$$

Proof. According to Theorem 2

$$EN_n^2 = 1 + \sum_{k=1}^{n-1} w_{k,n} (2EN_k + 1).$$

From Corollary 1 we know that $EN_k = \frac{2}{3} H_k + \frac{1}{3}$. Hence

$$EN_n^2 = 1 + \sum_{k=1}^{n-1} w_{k,n} \left(\frac{4}{3} H_k + \frac{5}{3} \right).$$

The proof of Corollary 1 shows that

$$\sum_{k=1}^{n-1} w_{k,n} = \frac{2}{3} H_n - \frac{2}{3}.$$

Furthermore

$$\begin{aligned}\sum_{k=1}^{n-1} w_{k,n} H_k &= \sum_{k=1}^{n-1} 2k(k+1) \sum_{i=k+1}^{n} \frac{1}{(i^2-1)i^2} \sum_{j=1}^{k} \frac{1}{j} \\ &= 2\sum_{i=2}^{n} \frac{1}{(i^2-1)i^2} \sum_{k=1}^{i-1} k(k+1) \sum_{j=1}^{k} \frac{1}{j} \\ &= 2\sum_{i=2}^{n} \frac{1}{(i^2-1)i^2} \sum_{j=1}^{i-1} \frac{1}{j} \sum_{k=j}^{i-1} k(k+1) \\ &= \frac{2}{3} H_n^{(1,1)} - \frac{2}{9} H_n - \frac{11}{18} + \frac{1}{3}\frac{1}{n+1} \\ &= \frac{1}{3} H_n^2 - \frac{2}{9} H_n + \frac{1}{3} H_n^{(2)} - \frac{11}{18} + \frac{1}{3}\frac{1}{n+1}.\end{aligned}$$

Thus

$$EN_n^2 = \frac{4}{9} H_n^2 + \frac{22}{27} H_n + \frac{4}{9} H_n^{(2)} - \frac{25}{27} + \frac{4}{9}\frac{1}{n+1},$$

and Corollary 1 implies the stated expression for $\operatorname{var} N_n$. □

Corollary 3. *The third moment of N_n is given by*

$$\begin{aligned}EN_n^3 = &\frac{8}{27} H_n^3 + \frac{32}{27} H_n^2 - \frac{106}{81} H_n + \frac{8}{9} H_n H_n^{(2)} + \frac{32}{27} H_n^{(2)} + \frac{16}{27} H_n^{(3)} \\ &- \frac{16}{9} H_n^{(2,1)} + \frac{91}{81} - \frac{8}{9}\frac{1}{n} H_n - \frac{16}{27}\frac{1}{n+1}\end{aligned}$$

and the third cumulant of N_n by

$$\begin{aligned}\kappa_3(N_n) = &\frac{14}{81} H_n + \frac{20}{27} H_n^{(2)} + \frac{16}{27} H_n^{(3)} - \frac{16}{9} H_n^{(2,1)} + \frac{172}{81} \\ &- \frac{8}{9}\left(\frac{1}{n} + \frac{1}{n+1}\right) H_n - \frac{28}{27}\frac{1}{n+1}.\end{aligned}$$

Proof. According to Theorem 2

$$EN_n^3 = 1 + \sum_{k=1}^{n-1} w_{k,n} (3EN_k^2 + 3EN_k + 1).$$

Elementary calculations like the ones in the proofs of Corollaries 1 and 2 yield the stated expression for EN_n^3. Recalling that $\kappa_3(N_n) = EN_n^3 - 3EN_n^2 EN_n + 2(EN_n)^3$ the expression for $\kappa_3(N_n)$ follows. □

Corollary 4. *The fourth moment of N_n is given by*

$$\begin{aligned} EN_n^4 &= \frac{16}{81}H_n^4 + \frac{112}{81}H_n^3 - \frac{148}{243}H_n^2 + \frac{32}{27}H_n^2H_n^{(2)} + \frac{914}{729}H_n + \frac{112}{27}H_nH_n^{(2)} \\ &\quad + \frac{128}{81}H_nH_n^{(3)} - \frac{128}{27}H_nH_n^{(2,1)} - \frac{16}{9}\big(H_n^{(2)}\big)^2 - \frac{148}{243}H_n^{(2)} + \frac{224}{81}H_n^{(3)} \\ &\quad - \frac{32}{27}H_n^{(4)} - \frac{224}{27}H_n^{(2,1)} - \frac{128}{27}H_n^{(3,1)} + \frac{256}{27}H_n^{(2,1,1)} - \frac{803}{729} \\ &\quad + \frac{32}{27}\frac{1}{n+1}H_n^2 + \frac{16}{27}\frac{1}{n}H_n + \frac{32}{27}\frac{1}{n+1}H_n^{(2)} + \frac{140}{243}\frac{1}{n+1} \end{aligned}$$

and the fourth cumulant of N_n by

$$\begin{aligned} \kappa_4(N_n) &= \frac{62}{729}H_n - \frac{64}{27}\big(H_n^{(2)}\big)^2 + \frac{212}{243}H_n^{(2)} + \frac{160}{81}H_n^{(3)} - \frac{32}{27}H_n^{(4)} - \frac{160}{27}H_n^{(2,1)} \\ &\quad - \frac{128}{27}H_n^{(3,1)} + \frac{256}{27}H_n^{(2,1,1)} - \frac{4724}{729} + \frac{64}{27}\left(\frac{1}{n} + \frac{1}{n+1}\right)H_n^2 \\ &\quad + \frac{16}{9}\left(\frac{1}{n} + \frac{1}{n+1}\right)H_n + \frac{1076}{243}\frac{1}{n+1} - \frac{16}{27}\frac{1}{(n+1)^2}. \end{aligned}$$

Proof. The calculations leading to EN_n^4 are analogous to the proofs of the preceding corollaries; $\kappa_4(N_n)$ follows according to $\kappa_4(N_n) = EN_n^4 - 4EN_n^3EN_n - 3(EN_n^2)^2 + 12EN_n^2(EN_n)^2 - 6(EN_n)^4$. □

Theorem 3. *For any $m \in \mathbb{N}$ the asymptotic behaviour of the m-th moment of N_n is given by*

$$EN_n^m = \left(\frac{2}{3}\log n\right)^m + O\big(\log^{m-1} n\big)$$

as n tends to infinity.

Proof. In order to derive the claimed expression from Theorem 2, we need precise information about the asymptotic behaviour of $w_{k,n}$. The required information is provided by the formula

$$\begin{aligned} \sum_{i=k+1}^{n} \frac{1}{(i^2-1)i^2} &= \sum_{p=3}^{r} \frac{(p-2)!}{p}\left(\prod_{q=0}^{p-1}\frac{1}{k+q} - \prod_{q=0}^{p-1}\frac{1}{n+q}\right) \\ &\quad + \sum_{i=k+1}^{n} \frac{(r-1)!}{(i^2-1)i^2}\prod_{q=2}^{r-1}\frac{1}{i+q} \qquad (r \geq 2). \end{aligned}$$

For $r = 2$ the right hand side of the formula reduces to the left hand side. For $r \geq 3$ it consists of the $r-2$ summands corresponding to $p = 3, \dots, r$, the asymptotic behaviour of which is obvious, and a remainder term, which has the same structure as the left hand side, but is much smaller.

Assume that the formula has been derived for some $r \geq 2$. Replacing in the denominator of the second sum the value i^2 by $i(i+r)$, a slightly smaller sum arises, which can be evaluated easily:

$$\begin{aligned}
&\sum_{i=k+1}^{n} \frac{(r-1)!}{(i-1)i(i+1)(i+2)\ldots(i+r-1)(i+r)} \\
&= (r-1)! \sum_{i=k+1}^{n} \frac{1}{(r+1)!}\left(\frac{\binom{r+1}{0}}{i-1} - \frac{\binom{r+1}{1}}{i} + \cdots + (-1)^{r+1}\frac{\binom{r+1}{r+1}}{i+r}\right) \\
&= \frac{(r-1)!}{r+1}\left(\frac{1}{r!}\left(\frac{\binom{r}{0}}{k} - \frac{\binom{r}{1}}{k+1} + \cdots + (-1)^{r}\frac{\binom{r}{r}}{k+r}\right)\right. \\
&\qquad\qquad \left. - \frac{1}{r!}\left(\frac{\binom{r}{0}}{n} - \frac{\binom{r}{1}}{n+1} + \cdots + (-1)^{r}\frac{\binom{r}{r}}{n+r}\right)\right) \\
&= \frac{(r-1)!}{r+1}\left(\prod_{q=0}^{r}\frac{1}{k+q} - \prod_{q=0}^{r}\frac{1}{n+q}\right).
\end{aligned}$$

The difference

$$\sum_{i=k+1}^{n} \frac{(r-1)!}{(i^2-1)i^2}\prod_{q=2}^{r-1}\frac{1}{i+q} - \sum_{i=k+1}^{n}\frac{(r-1)!}{(i-1)\ldots(i+r)}$$

is just

$$\sum_{i=k+1}^{n}\frac{r!}{(i^2-1)i^2}\prod_{q=2}^{r}\frac{1}{i+q},$$

proving the formula. An estimate of the remainder term is given by

$$\begin{aligned}
0 &< \sum_{i=k+1}^{n}\frac{(r-1)!}{(i^2-1)i^2}\prod_{q=2}^{r-1}\frac{1}{i+q} < \sum_{i=k+1}^{n}\frac{(r-1)!}{i^{r+2}} \\
&< (r-1)!\int_k^n \frac{1}{x^{r+2}}\mathrm{d}x = \frac{(r-1)!}{r+1}\left(\frac{1}{k^{r+1}} - \frac{1}{n^{r+1}}\right)
\end{aligned}$$

for $1 \leq k \leq n-1$ and $r \geq 3$.

Corollary 1 implies Theorem 3 in the case $m = 1$. Assume that for $m \geq 2$ Theorem 3 has been proved for the moments up to order $m-1$. Then we have

$$E(N_k+1)^m - EN_k^m = \sum_{j=0}^{m-1}\binom{m}{j}EN_k^j = m\left(\frac{2}{3}\log k\right)^{m-1} + O\left(\log^{m-2} k\right)$$

as k tends to infinity. Now we apply the formula with $r = 3$. We immediately see that

$$\sum_{k=1}^{n-1}\frac{2}{3}\frac{1}{k+2}\left(E(N_k+1)^m - EN_k^m\right) = \left(\frac{2}{3}\log n\right)^m + O\left(\log^{m-1} n\right)$$

and that

$$\sum_{k=1}^{n-1}\frac{2}{3}\frac{k(k+1)}{n(n+1)(n+2)}\big(E(N_k+1)^m-EN_k^m\big)=\frac{2}{9}m\Big(\frac{2}{3}\log n\Big)^{m-1}+O\big(\log^{m-2}n\big)$$

as n tends to infinity. The estimate of the remainder term yields that

$$\sum_{k=1}^{n-1}4k(k+1)\big(E(N_k+1)^m-EN_k^m\big)\sum_{i=k+1}^{n}\frac{1}{(i^2-1)i^2(i+2)}$$

tends to a constant. □

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