POISSON EQUATION & SCHAUDER ESTIMATES

Lecture 9.

Solving $\Delta u = f$ on $\mathbb{R}^n$, for say $f \in S(\mathbb{R}^n)$, yields

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\,dy.$$

By the convolution properties: $f \in C^\infty \Rightarrow u \in C^\infty$ (i.e. $Du = D(\Gamma * f) = \Gamma * Df$.)

However if $f \in C^0$ then $u \notin C^2$! (in general).

e.g. ($f \in L^\infty$, but $D^2 u \notin L^\infty$)

Let $u(x,y) = |x| \cdot |y| \cdot \log(|x| + |y|)$.

Then for $x, y > 0$: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - \frac{2xy}{(x+y)^2} \in L^\infty$.

However, (for $x,y>0$)

$$\frac{\partial^2 u}{\partial x \partial y} = \log(x+y) + 1 - \frac{xy}{(x+y)^2} \notin L^\infty \quad (\leftarrow \text{unbounded at } 0)$$

e.g. ($f \in C^0$ but $D^2 u \notin C^0$) NB: there are no classical problems to this solution.

$$\Delta u = f(x) = \begin{cases} \dfrac{y^2 - x^2}{2|x|^2}\left(\dfrac{n+2}{(-\log|x|)^{1/2}} + \dfrac{1}{2(-\log|x|)^{3/2}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then f is continuous at the origin, however the solution

$$u(x) = (x^2 - y^2)(-\log|x|)^{1/2}$$

has $\frac{\partial^2 u}{\partial x^2} \to \infty$ as $x \to 0$. Therefore $u \notin C^2$.

1. Hölder continuous spaces

Def. We say $u \in C^0(\Omega)$ is Hölder continuous with exponent α if

$$[u]_{\alpha;\Omega} = \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \quad 0 < \alpha < 1,$$

is finite. We denote by $C^\alpha(\Omega)$, or $C^{0,\alpha}(\Omega)$, the set of all such functions and let $C^\alpha_{loc}(\Omega)$ denote the set of all $u \in C^\alpha(\Omega')$ for $\Omega' \subset\subset \Omega$.

NB: When $\alpha = 1$, the function u is said to be Lipschitz continuous.

e.g. $u(x) = |x|^\beta$ on $(0,1)$ with $0 < \beta \leq 1$ has

(i) $u \in C^\alpha$ for $0 < \alpha \leq \beta$;

(ii) $u \notin C^\alpha$ for $\alpha > \beta$.

Remark. The space $C^0(\Omega)$ is a Banach space with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} = \sup_\Omega |u| + [u]_{\alpha;\Omega}.$$

NB: $C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$ is merely a metric space

Likewise, we say $u \in C^{k,\alpha}(\Omega)$, for $k \in \mathbb{N}$, if $D^\beta u \in C^{0,\alpha}(\Omega)$ for all $|\beta| = k$.

By setting

$$|D^k u|_{0;\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|$$

$$[D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}$$

We can define the norms

$$\begin{cases} \|u\|_{C^k(\Omega)} = \sum_{j=0}^{k} |D^j u|_{0;\Omega} \\ \|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^k(\Omega)} + [D^k u]_{\alpha;\Omega} \end{cases}.$$

e.g. $u(x) = \begin{cases} \frac{-1}{\log|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ has $u \in C^0$ but $u \notin C^\alpha(-\frac{1}{2}, \frac{1}{2})$ for any $0 < \alpha \leq 1$.

e.g. $u_\lambda(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^{k\lambda}}$, $0 < \lambda \leq 1$, has $u_\lambda \in C^\alpha(0,\pi)$, $\forall\, 0 < \alpha < \lambda$ but $u_\lambda \notin C^1$.

Remark. The inclusions

$$C^0(\Omega) \supset C^{0,\alpha}(\Omega) \supset C^1(\Omega)$$

for $0 < \alpha < 1$ are strict.

Theorem (Rademacher)

If $u \in C^{0,1}(\Omega)$ then u is differentiable a.e. in Ω.

(i.e. the set of points at which u is not differentiable is a set of Lebesgue measure zero)

2. Schauder estimates

Def. A modulus of continuity (MOC) is a continuous real-extended valued function

$$\omega : [0,\infty] \to [0,\infty]$$

that vanishes at the origin, i.e. $\lim_{t \searrow 0} \omega(t) = \omega(0) = 0$.

NB: If $\omega(t)$ is decreasing, define $\omega^+(t) := \sup_{s \leq t} \omega(s)$ so that $\omega^+ \geq \omega$. Then the new MOC ω^+ is non-decreasing.

(Hence we can take MOC to be non-decreasing w.l.o.g)

Remark. The MOC measures quantitively both the pointwise continuity of, and the uniform continuity of, maps between metric spaces.

e.g. If $f : X \to Y$ between metric spaces, then f is said to admit ω as a MOC if

$$d(f(x), f(y)) \leq \omega(d(x,y)), \quad \forall x, y \in X.$$

Proposition. f is uniformly continuous if and only if it admits a MOC.

Def. A MOC ω is said to satisfy the <u>Dini condition</u> if

$$\int_0^1 \frac{\omega(t)}{t}\,dt < +\infty .$$

Now consider the Poisson equation

$$\Delta u = f \quad \text{in } B_1(0) \subset \mathbb{R}^n \qquad \text{—} \ (*)$$

Suppose f satisfies the Dini condition with MOC

$$\omega_f(t) = \sup\{ |f(x) - f(y)| : |x-y| < t \}.$$

Given this we obtain the following result:

<u>Theorem (Xu-Jia Wang, 2006)</u>

If u satisfies $(*)$, then for $x, y \in B_{1/2}(0)$,

$$|D^2u(x) - D^2u(y)| \le C\left[d \sup_{B_1} |u| + \int_0^d \frac{\omega_f(t)}{t}\,dt + d \int_d^1 \frac{\omega_f(t)}{t^2}\,dt \right] \quad \text{—} \ (A)$$

where $d = |x-y|$ and $C = C(n) > 0$.

Moreover, if $f \in C^{\alpha}(B_1)$ then

$$\begin{cases} \|u\|_{2,\alpha;B_{1/2}} \le C\left(\sup_{B_1} |u| + \dfrac{1}{\alpha(1-\alpha)} \|f\|_{0,\alpha;B_1} \right), & \text{if } 0<\alpha<1 \\ |D^2u(x) - D^2u(y)| \le C\left(d \sup_{B_1} |u| + |\log d|\, \|f\|_{0,1;B_1} \right), & \text{if } \alpha = 1 \end{cases}$$

NB: for the first condition we have $\omega(t) = [f]_\alpha\, t^\alpha$ and for the second, $\omega(t) = \mathrm{Lip}(f)\, t$.